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6ème Cours
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The Curve

Recall: $P = \bigoplus_{d \geq 0} B^{\varphi = \pi^d}$ $\qquad B = \mathcal{O}(Y)$ $= $ E-Frechet algebra

$X = \mathrm{Proj}(P)$

We want to prove that X is a Dedekind scheme.

* Case F alg. closed (the case of any perfectoid F is treated by Galois descent from $\widehat{\overline{F}}$ to F).

Last time we proved that P is graded factorial with irreducible elements of degree 1.

$\Rightarrow \forall t \in P_1 \setminus \{0\}$ $\quad P[\frac{1}{t}]_0 = B[\frac{1}{t}]^{\varphi = Id}$

is factorial with irreducible elements

$$\left\{ \frac{t'}{t} \;/\; t' \in P_1 \setminus E.t \right\}$$

The fundamental exact sequence

Prop: Soient $t_1, \dots, t_d \in P_1 \setminus \{0\}$ and $y_1, \dots, y_d \in |Y|^{cl}$ associated i.e. $div(t_i) = \sum_{n \in \mathbb{Z}} [\varphi^n(y_i)]$. Let $y_i = V(a_i)$, $a_i \in A$ primitive degree 1. Then the sequence

$$0 \to E.\prod_{i=1}^{d} t_i \longrightarrow \underbrace{B^{\varphi = \pi^d}}_{P_d} \longrightarrow B/B_{a_1 \dots a_d} \to 0$$

est exacte.

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Ex: $t \in P_1$

$$0 \to E \cdot t^d \to B^{\varphi = \pi^d} \to B^+_{dR,y} / t^d B^+_{dR,y} \to 0$$

if $\mathrm{div}(t) = \sum_{n \in \mathbb{Z}} [\varphi^n(y)]$

Proof: Exactness in the middle:

If $f \in B^{\varphi = \pi^d} \setminus \{0\}$ satisfies $f \in B a_1 \dots a_d$

$$\Rightarrow \mathrm{div}(f) \geq \sum_{i=1}^{d} \underbrace{[y_i]}_{\mathrm{div}(a_i)}$$

$$\Rightarrow \mathrm{div}(f) \geq \sum_{n \in \mathbb{Z}} \varphi^n\left(\sum_{i=1}^{d} [y_i]\right)$$

$\mathrm{div}(f)$ φ-invariant

$$= \sum_{i=1}^{d} \mathrm{div}(t_i)$$

$$= \mathrm{div}\left(\prod_{i=1}^{d} t_i\right)$$

$$\Rightarrow f = x \cdot \prod_{i=1}^{d} t_i \quad , \quad x \in B$$

But $\varphi(f) = \pi^d f$ and $\varphi(t_i) = \pi t_i \Rightarrow x \in B^{\varphi = Id} = E$.

<u>Surjectivity</u>: By dévissage it suffices to treat the case $d = 1$.

Case $E = \mathbb{Q}_p$ (Idem for E general with Lubin-Tate)

$$B^{\varphi = p} \xrightarrow{\theta} C = C_y \text{ where } y = (a)$$

$\varepsilon \in 1 + m_F$, $\log([\varepsilon]) \in B^{\varphi = p}$ and then

$$\theta(\log([\varepsilon])) = \log(\underbrace{\theta([\varepsilon])}_{\varepsilon^\# \in 1 + m_C})$$

C alg. closed $\Rightarrow$ $\log : 1 + m_C \twoheadrightarrow C$ is surjective

(if $x \in C$, for $n >> 0$ $p^n x \in$ domain of convergence of $\exp$
then if $z \in 1 + m_C$ verifies $z^{p^n} = \exp(p^n x)$, $\log z = x$)

Let $x \in C$, $x = \log(z)$ with $z \in 1 + m_C$

C alg. closed $\Rightarrow \exists\, \varepsilon \in C^\flat$ s.t. $\varepsilon^\# = z$. Via $F \xrightarrow{\sim} C^\flat$

$\varepsilon \in 1 + m_F$ and $x = \log\big(\underbrace{\theta([\varepsilon])}_{\varepsilon^\#}\big) = \theta\big(\log([\varepsilon])\big)$ $\square$

<u>Rem</u>: We're going to use the fundamental exact sequence to prove that $X = \mathrm{Proj}(P)$ is a curve. Reciprocally, once the curve is constructed we can find back the fundamental exact sequence

X curve $\mathcal{O}_X(1)$ line bundle

$P_d = H^0(X, \mathcal{O}_X(d))$

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\times \prod_{i=1}^{d} t_i} \mathcal{O}_X(d) \to \mathcal{F} \to 0$$

$\uparrow$ torsion coherent

exact sequence of coherent sheaves$/X$

+ application of $H^0(X,-)$ and use $H^1(X, \mathcal{O}_X) = 0$.

<u>Corollary</u>: $t \in P_1 \setminus \{0\}$. $C = C_y$ where $\mathrm{div}(t) = \sum_{n \in \mathbb{Z}} [\varphi^n(y)]$

There is an isomorphism of E-graded algebras

$$P/tP \xrightarrow{\sim} \{ f \in C[T] \mid f(0) \in E \}$$

$$\begin{array}{l} \| \\ E \oplus \bigoplus_{d \geq 1} P_d / t P_{d-1} \end{array}$$

$$\begin{array}{l} x \bmod t P_{d-1} \longmapsto \partial_y(x) \cdot T^d \\ \cap \\ P_d \end{array}$$

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Theo: (1) P graded factorial + irred. elements of deg. 1

(2) If $t \in P_1 \setminus \{0\}$ $\quad P/tP \xrightarrow{\sim} \underbrace{\{f \in C_y[T] \,/\, f(0) \in E\}}_{D_y}$

$y \in |Y|^{cl}$, $t(y) = 0$. $\quad$ graded iso.

Th: $X = \mathrm{Proj}(P)$ is a Dedekind scheme

If $t \in P_1 \setminus \{0\}$, $V^+(t) = \{\infty_t\}$ closed point of X

and $X \setminus \{\infty_t\} = \mathrm{Spec}\left(\underbrace{B\left[\frac{1}{t}\right]^{\varphi = Id}}_{P.I.D.}\right)$

Proof: $V^+(t) = \mathrm{Proj}(P/tP) \simeq \mathrm{Proj}(D_y)$

Easy computation: the only prime homogenous ideal not containing the augmentation ideal of D_y is the zero ideal.

$\Rightarrow V^+(t) =$ only one point.

Moreover $\mathrm{Proj}(D_y) = \mathrm{Spec}(D_y[\frac{1}{T}]_0) = \mathrm{Spec}(C_y)$

[the only point lies in $D^+(T)$]

via $C_y \xrightarrow{\sim} D_y[\frac{1}{T}]_0$

$z \mapsto \frac{z}{T}$

$\Rightarrow V^+(t) = \{\infty_t\}$ closed point.

$*$ $B_e = B[\frac{1}{t}]^{\varphi = \mathrm{Id}}$ factorial with irreducible elements $\{\frac{t'}{t} / t' \in P_1 \setminus E.t\}$

Moreover $B_e / (\frac{t'}{t}) = P/t'P\,[\frac{1}{E}]_0$ where $E = t \bmod t' \in (P/t'P)_1$

$\simeq C_{y'}$ if $t'(y') = 0$.

[if $a \in (D_{y'})_1$ $D_{y'}[\frac{1}{a}]_0 = C_{y'}$.

$\Rightarrow$ The maximal ideal generated by any irreducible element is maximal

$\Rightarrow B_e$ is principal. $\square$

Prop: $f \in E(X)^\times \quad \deg(\operatorname{div} f) = 0$

proof. $t \in P_1 \setminus \{0\} \qquad E(X) = \operatorname{Frac}(B_e)\,, \quad B_e = B[\tfrac{1}{t}]^{\varphi = \mathrm{Id}}$

It suffices to prove it for $f \in B_e \setminus \{0\}$

If $f \in E^\times$ this is clear since then $\operatorname{div} f = 0$.

If not, $\exists d \geq 1 \qquad f = \dfrac{t'_1 \dots t'_d}{t^d}$

$$t'_1, \dots, t'_d \in P_1 \setminus \{0\}$$

$$\operatorname{div} f = \sum_{i=1}^{d} [\infty_{t'_i}] - d[\infty_t] \Rightarrow \text{degree } 0 \quad \square$$

Prop: If $\mathrm{div}(t) = \sum_{n \in \mathbb{Z}} [\varphi^n(y)]$

$$\widehat{\mathcal{O}}_{X,\infty_t} \xrightarrow{\sim} B^+_{dR,y}$$

Proof:

$$\mathcal{O}_{X,\infty_t} = \left\{ \frac{a}{b} \;\middle|\; \begin{array}{l} a, b \in P \text{ homogeneous same} \\ \text{degree and } b \notin tP \end{array} \right\} \subset \mathrm{Frac}(P)$$

$$\mathrm{Frac}(P) \supset \mathrm{Frac}(B) \supset B_{dR,y}$$

$$\mathcal{O}_{X,\infty_t} \longrightarrow B^+_{dR,y} \longrightarrow B_{dR,y}$$

since $b \notin P.t \iff b(y) \neq 0$.

$\mathcal{O}_{X,\infty_t} \subset B^+_{dR,y}$ inclusion of D.V.R.

Same uniformizing element $\frac{t}{t'}$ where $t' \in P_1 \setminus E.t$

unif. element of $B^+_{dR,y}$ since $\mathrm{ord}_y\left(\frac{t}{t'}\right) = 1$.

+ iso. at the level of residue fields $\Rightarrow$ iso. on completion $\square$

Background on vector bundles / curves

X Dedekind scheme integral

$\infty \in |X|$ closed point - η generic point

$K = \mathcal{O}_{X,\eta}$

Suppose moreover $X \setminus \{\infty\} = \mathrm{Spec}(A)$ affine

A = Dedekind ring

$t \in \underbrace{\mathcal{O}_{X,\infty}}_{\text{D.V.R.}}$ uniformizing element

<u>Vector bundles</u>: Bun_X = locally free $\mathcal{O}_X$-modules of finite rank

$\mathcal{C} = \{ (M, W, u) \mid$ M = ~~B~~ projective ~~A~~-module of f.t.

$W = \widehat{\mathcal{O}}_{X,\infty}$-module free of f.t.

$u : M \otimes_B \widehat{\mathcal{O}}_{X,\infty}[\frac{1}{t}] \xrightarrow{\sim} W[\frac{1}{t}]$ $\}$

particular case of Beauville–Laszlo.

Then $\left[\begin{array}{l} \mathrm{Fil}_X \xrightarrow{\sim} \mathcal{C} \\ \mathcal{E} \longmapsto \left(\Gamma(X \setminus \infty, \mathcal{E}), \widehat{\mathcal{E}}_\infty, \mathrm{can} \right) \end{array} \right]$

~~At~~ where "can" is derived from

$\Gamma(X \setminus \infty, \mathcal{E}) \hookrightarrow \mathcal{E}_\eta = \mathcal{E}_\infty[\frac{1}{t}]$

$\left[\underset{=}{\mathrm{Ex}}: \ \mathcal{O}_X(d[\infty]) \longleftrightarrow \left(A, t^{d} \widehat{\mathcal{O}}_{X,\infty}, \mathrm{can} \right) \right]$

Moreover if $\mathcal{E} \longleftrightarrow (M, W, u)$

$$R\Gamma(X,\mathcal{E}) \simeq \left[\overset{0}{M \oplus W} \longrightarrow \overset{+1}{W[\tfrac{1}{t}]} \right]$$

$$(m,w) \longmapsto u(m) - w$$

$$\Rightarrow \boxed{\begin{aligned} H^0(X,\mathcal{E}) &= u(M) \cap W \\ H^1(X,\mathcal{E}) &= W[\tfrac{1}{t}] \big/ W + u(M) \end{aligned}}$$

* Suppose now that X is "complete" i.e. we have a function $\deg : |X| \to \mathbb{N}_{\geq 1}$ satisfying

$$\forall f \in K^\times,\ \deg(\operatorname{div} f) = 0.$$

Suppose moreover that $\deg(\infty) = 1$.

$$\Longrightarrow H^0(X, \mathcal{O}_X) = \text{field} =: E \subset K$$

$$\begin{array}{c} \| \\ \{0\} \cup \{ f \in K^\times \,/\, \operatorname{div}(f) \geq 0 \} \end{array}$$

But $\operatorname{div}(f) \geq 0 \iff \operatorname{div}(f) = 0.$

$\uparrow \deg(\operatorname{div} f) = 0$

Then $\deg := -\operatorname{ord}_\infty : A \longrightarrow \mathbb{N} \cup \{-\infty\}$

$\uparrow$ valuation on K

$$E = A^{\deg \leq 0}$$

$$H^0(X, \mathcal{O}_X(d[\infty])) = A^{\deg \leq d}$$

$$H^1(X, \mathcal{O}_X(d[\infty])) = K \big/ E^{-d} \mathcal{O}_{X,\infty} + A$$

In particular $\boxed{H^1(X, \mathcal{O}_X) = K \big/ \mathcal{O}_{X,\infty} + A}$

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$H^1(X, \mathcal{O}_X) = 0 \iff \forall x, y \in A$ with $y \neq 0$, $\exists a \in A$ s.t. $\frac{x}{y} - a \in \mathcal{O}_{X,\infty}$

$\Updownarrow$ $\qquad\qquad\qquad\qquad \Updownarrow$

$(A, \deg)$ almost euclidean $\qquad \deg\left(\frac{x}{y} - a\right) \leq 0$

$\Updownarrow$

$\deg(x - ay) \leq \deg(y)$

* $H^1(X, \mathcal{O}_X(-\infty)) = K / \mathcal{O}_{X,\infty} + A$

$H^1(X, \mathcal{O}_X(-1)) = 0 \iff \forall x, y \in A,\ y \neq 0,\ \exists a \in A$

$\Updownarrow$ $\qquad\qquad \deg\left(\frac{x}{y} - a\right) < 0$

$(A, \deg)$ euclidean $\qquad \Updownarrow$

$\deg(x - ay) < \deg(y)$

* $\deg(\operatorname{div} f) = 0 \Rightarrow$ $\qquad \deg : \operatorname{Div}(X) \longrightarrow \mathbb{Z}$ factorizes

$\deg : \operatorname{Div}(X)/\sim \longrightarrow \mathbb{Z}$

$\qquad \|$

$\operatorname{Pic}(X)$

We then have

$$Cl(A) = \mathrm{Div}(\mathrm{Spec}(A))/\sim \;\xrightarrow{\sim}\; \mathrm{Div}^0(X)/\sim = \mathrm{Pic}^0(X)$$

$$[D] \longmapsto [D - \deg D \cdot [\infty]]$$

Thus $\Big[$ A is a P.I.D. $\iff$ $\mathrm{Pic}^0(X) = 0$

$\iff$ $\deg : \mathrm{Pic}(X) \xrightarrow{\sim} \mathbb{Z}$. $\Big]$